

LYAPUNOV EXPONENTS OF RANK 2-VARIATIONS OF HODGE STRUCTURES AND MODULAR EMBEDDINGS

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ABSTRACT. If the monodromy representation of a VHS over a hyperbolic curve stabilizes a rank two subspace, there is a single non-negative Lyapunov exponent associated with it. We derive an explicit formula using only the representation in the case when the monodromy is discrete.

1. INTRODUCTION

The Lyapunov exponents of a dynamical cocycle are usually hard to come by. The action of the Teichmüller geodesic flow on the relative cohomology bundle over the moduli space of curves \mathcal{M}_g is a striking exception, since much information can be obtained from a formula for the *sum of the non-negative Lyapunov exponents* originally discovered by Kontsevich and Zorich [KZ97] (see also [For02], [EKZ10]). It exploits a link between algebraic geometry and dynamical systems and expresses the sum as integrals over certain characteristic classes of \mathcal{M}_g .

Variants of this result are known to hold for subsets invariant under the flow such as Teichmüller curves, which are algebraic curves in \mathcal{M}_g isometrically embedded with respect to the Teichmüller metric. One can even replace the Teichmüller flow by the geodesic flow on an arbitrary hyperbolic curve \mathbb{H}/Γ (or more generally a ball quotient, see [KM12]) and an analogous formula will hold for the dynamical cocycle coming from the monodromy action of the fundamental group Γ on the cohomology of a family of curves $\phi : \mathcal{X} \rightarrow \mathbb{H}/\Gamma$ (or more generally on a variation of Hodge structures (VHS) of weight one).

In this paper, we focus on the situation when there exists a subbundle of rank two of such a relative cohomology bundle over a curve. It has only one non-negative Lyapunov exponent. Starting from the Kontsevich-Zorich formula, we show how to effectively compute this exponent only from the representation of the fundamental group.

Theorem 1.1. *Let $\phi : \mathcal{X} \rightarrow C$ be a family of curves over a non-compact algebraic curve $C = \mathbb{H}/\Gamma$, and suppose there exists a rank 2-submodule $V \subseteq H^1(\mathcal{X}_c, \mathbb{R})$ invariant under the monodromy action ρ_V of $\Gamma = \pi_1(C, c)$ such that $\rho_V(\Gamma)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.*

Then the non-negative Lyapunov exponent associated with V is 0 if Γ acts as a finite group and is otherwise given by

$$\lambda = \frac{\mathrm{vol}(\mathbb{H}/\rho_V(\Gamma))}{\mathrm{vol}(\mathbb{H}/\Gamma)} \sum_{\Gamma_0 \leq \Gamma} (\Delta_0 : \rho_V(\Gamma_0))$$

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where Δ_0 is a fixed parabolic subgroup of $\rho_V(\Gamma)$ and the sum runs over a system of representatives of conjugacy classes of maximal parabolic subgroups Γ_0 of Γ , whose generator is mapped to $\Delta_0 \setminus \{\pm I\}$.

If the relative cohomology of the family of curves over a (finite cover of a) Teichmüller curve has a rank-2 subbundle invariant under the flow and defined over \mathbb{Q} , we can compute the associated Lyapunov exponent from the monodromy representation of the affine group. We carry this out for an example, where even a complete splitting into 2-dimensional pieces is found.

Proposition 1.2. *The Lyapunov spectrum of the Teichmüller curve generated by the square-tiled surface $(X, \omega) \in \Omega\mathcal{M}_4(2, 2, 2)^{\text{odd}}$ given by*

$$r = (1, 4, 7)(2, 3, 5, 6, 8, 9) \quad \text{and} \quad u = (1, 6, 8, 7, 3, 2)(4, 9, 5),$$

(see Figure 2) is $1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -1$.

Besides the Kontsevich-Zorich formula, the proof of Theorem 1.1 makes use of the period map $p : \mathbb{H} \rightarrow \mathbb{H}$ from the universal covering of C to the classifying space of Hodge structures of weight one on a two-dimensional \mathbb{R} -vector space. This map is equivariant for the two actions of Γ and, in case $\rho_V(\Gamma) \subseteq \text{SL}_2(\mathbb{R})$ is discrete and not finite, descends to a holomorphic map \bar{p} between algebraic curves. The main observation is that the line bundle is a pullback of the cotangent bundle by \bar{p} , and that one can compute the degree of \bar{p} by looking at the cusps.

From an abstract point of view, Theorem 1.1 deals with pairs (p, ρ) of a homomorphism $\rho : \Gamma \rightarrow \text{SL}_2(\mathbb{R})$ from a cofinite Fuchsian group Γ and a holomorphic map $p : \mathbb{H} \rightarrow \mathbb{H}$ equivariant for the actions of Γ and $\rho(\Gamma)$, which we call *modular embeddings*. We show that these are rigid in the sense that p and ρ almost uniquely determine each other, a fact that has been remarked in [McM03] for Teichmüller curves in genus 2. Moreover, we introduce the notion of (*weak*) *commensurability* of two modular embeddings (they must agree (up to conjugation) on some finite index subgroup). It follows that the Lyapunov exponent of a modular embedding is a weak commensurability invariant. We also investigate the commensurator of a modular embedding and show that it contains Γ as a subgroup of finite index if ρ has a non-trivial kernel.

Every rational number in $[0, 1]$ is a Lyapunov exponent of a Teichmüller curve in \mathcal{M}_g as can be deduced e.g. from [BM10, Theorem 4.5], [EKZ11, Prop. 2] or [Wri12a, Theorem 1.3]. However the denominator of the rational numbers that can be reached depends on g . In Proposition 5.7, we combine the discussion of modular embeddings with Theorem 1.1 to obtain the same result by pulling back the universal family of elliptic curves via a complicated map. The resulting family will of course not map to a Teichmüller curve in moduli space.

References. Previously, period maps have been used to compute the individual Lyapunov exponents of Teichmüller curves coming from abelian covers of \mathbb{P}^1 [Wri12b]. In this situation, the period maps are Schwarz triangle maps, the monodromy is a possibly indiscrete triangle group, and the Lyapunov exponents are quotients of areas of hyperbolic triangles. Other examples, where individual Lyapunov exponents have been obtained by computing the degrees of line bundles, are the Veech-Ward-Bouw-Möller-Teichmüller curves [BM10], [Wri12b], cyclic covers of \mathbb{P}^1 [EKZ11] and more generally Deligne-Mostow ball quotients [KM12].

Modular embeddings of \mathbb{H} into a product \mathbb{H}^k have been studied e.g. in [CW90] for the action of a Schwarz triangle group on the left and the direct product of its Galois conjugates on the right (where k is the degree of the trace field over \mathbb{Q}) or for non-arithmetic Teichmüller curves in [Möl06], [McM03] for the action of the Veech group and its Galois conjugates. Our definition relates to theirs (for $k = 2$) if one considers the (id, ρ) -equivariant embedding $\mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$, $z \mapsto (z, p(z))$.

Structure of the paper. The paper is organized as follows. Section 2 contains the necessary background on Teichmüller curves, variations of Hodge structures and Lyapunov exponents. Section 3 contains the proof of Theorem 1.1. In Section 4, we discuss an algorithmic approach to the computation of Lyapunov exponents and present two examples, the one stated above being among them. Finally, in Section 5, we discuss various properties of modular embeddings.

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2. BACKGROUND

In this section, we recall the concept of a variation of Hodge structures, the definition of the period map and the Kontsevich-Zorich formula and then specialize to the case of Teichmüller curves.

2.1. Variations of Hodge structures. Let C be a smooth algebraic curve over \mathbb{C} , embedded in a projective curve \overline{C} . A family $\phi : \mathcal{X} \rightarrow C$ of smooth curves defines a \mathbb{Z} -local system $\mathbb{V} = R^1\phi_*\mathbb{Z}$ on C , whose associated holomorphic vector bundle comes with a holomorphic subbundle $\mathbb{V}^{1,0} \subset \mathbb{V} \otimes_{\mathbb{Z}} \mathcal{O}_C$, inducing the Hodge decomposition of the cohomology in each fiber $\mathcal{X}_c = \phi^{-1}(c)$. This object, which is actually the family of Jacobians associated with ϕ , has been abstractly studied under the name *variation of Hodge structures of weight 1*; these consist of a K -local system \mathbb{V} on C (K a noetherian subring of \mathbb{R}) and a holomorphic subbundle $\mathbb{V}^{1,0} \subset \mathbb{V} \otimes_K \mathcal{O}_C$ inducing a Hodge structure in each fiber.

Important for the study of variations of Hodge structures is the presence of a *polarization*, which in our case is the intersection pairing on (co-)homology. It is defined as a locally constant alternating bilinear form $Q : \mathbb{V} \otimes \mathbb{V} \rightarrow K$ such that its \mathbb{C} -linear extension satisfies the Riemann bilinear relations $Q(\mathbb{V}^{1,0}, \mathbb{V}^{1,0}) = 0$ and $iQ(v, \overline{v}) > 0$ for non-zero $v \in \mathbb{V}^{1,0}$. The norm $\| \cdot \|$ associated with the positive definite hermitian form $\frac{i}{2}Q(v, \overline{v})$ on $\mathbb{V}^{1,0}$ on $\mathbb{V}_{\mathbb{R}}$ by

$$\|v\| = \frac{i}{2}Q(v^{1,0}, \overline{v^{1,0}})$$

(where $v^{1,0}$ denotes the projection of $v \in \mathbb{V}_{\mathbb{R}}$ to $\mathbb{V}^{1,0}$) is called Hodge norm. In the following, we write VHS for “polarized variation of Hodge structures of weight 1”.

By a *local monodromy* of \mathbb{V} about a puncture $c \in \overline{C} \setminus C$, we shall understand the action of a small loop about c on the fiber \mathbb{V}_c of a nearby point c . If K is a number field, then by a Theorem of Borel, these transformations are always quasi-unipotent. If they are unipotent, then there is a canonical extension due to Deligne of $\mathbb{V} \otimes_K \mathcal{O}_C$ to a vector bundle \mathcal{V} on \overline{C} . The extension of the $(1, 0)$ -part inside \mathcal{V} will be denoted $\mathcal{V}^{1,0}$.

The (global) *monodromy* is the linear representation of $\pi_1(C, c)$ on \mathbb{V}_c associated with the local system \mathbb{V} (and uniquely determined up to conjugation).

A standard reference for variations of Hodge structures is [CMSP03].

2.1.1. Decomposition of a VHS. By the work of Deligne, the category of \mathbb{C} -VHS on a quasiprojective base is semisimple. More precisely [Del87], if \mathbb{V} is a VHS on a smooth quasiprojective algebraic variety X over \mathbb{C} , then

$$(2.1) \quad \mathbb{V} \cong \bigoplus_i \mathbb{V}_i \otimes W_i$$

where \mathbb{V}_i are irreducible local systems, $\mathbb{V}_i \not\cong \mathbb{V}_j$ for $i \neq j$ and W_i are complex vector spaces. Moreover, each \mathbb{V}_i carries a VHS unique up to shifting of the bigrading, such that (2.1) is an isomorphism of VHS.

2.1.2. The period map and the period domain. Let $c \in C$ be a base point and let \mathbb{V} be a VHS on C . The underlying local system corresponds to the monodromy action of $\pi_1(C, c)$ on the fiber \mathbb{V}_c by continuation of local sections along paths. The distinguished subspace $\mathbb{V}_c^{1,0}$ of the Hodge filtration will be moved by this action; this movement is recorded by the *period map* $p : \tilde{C} \rightarrow \text{Per}(\mathbb{V}_c)$, which is a holomorphic map from the universal cover $u : \tilde{C} \rightarrow C$ to the *period domain* $\text{Per}(\mathbb{V}_c)$, the classifying space of polarized Hodge structures that can be put on \mathbb{V}_c .

The period map can be described in the following way: On \tilde{C} , the local system can be globally trivialized by the constant sheaf V of fiber \mathbb{V}_c and the inclusion $u^* \mathbb{V}^{1,0} \rightarrow V$ yields for every point $z \in \tilde{C}$ a Hodge structure on $V_z \cong \mathbb{V}_c$, thus a point $p(z) \in \text{Per}(\mathbb{V}_c)$. The fact that $u^* \mathbb{V}^{1,0} \rightarrow V$ is an inclusion of sheaves with $\pi_1(C, c)$ -action, corresponds to the map p being equivariant with respect to the action of $\pi_1(C, c)$ on \tilde{C} by deck transformations and on $\text{Per}(\mathbb{V}_c)$ by the monodromy action.

In the case of an \mathbb{R} -VHS of weight 1 and rank $2k$, $\text{Per}(\mathbb{V}_c) \cong \mathbb{H}_k$, the Siegel upper halfspace of dimension k and the monodromy is a representation of $\pi_1(C, c)$ into $\text{Sp}_{2k}(\mathbb{R})$.

A VHS \mathbb{V} on a curve C is called *uniformizing* if its period map is biholomorphic. In this case, $(\mathcal{V}^{1,0})^{\otimes 2} \cong \Omega_{\tilde{C}}^1(\log S)$ where $S = \tilde{C} \setminus C$ is the finite set of cusps. This isomorphism is given by the Kodaira-Spencer map, the only graded piece of the Gauß-Manin connection.

In particular, there is a tautological uniformizing VHS on each period domain and each VHS is equal to the pullback of a tautological VHS on its period domain via the period map. We sketch this for a rank 2-VHS, i.e. $k = 1$.

Suppose we are given a holomorphic map $p : \tilde{C} \rightarrow \mathbb{H}$ from the universal cover of a curve C , together with a group homomorphism $\rho : \pi_1(C, c) \rightarrow \text{Sp}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})$. The trivial bundle $\tilde{C} \times \mathbb{R}^2 \rightarrow \tilde{C}$ acquires a π_1 -action by

$$(z, v) \mapsto (\gamma(z), \rho(\gamma)(v)), \quad \gamma \in \pi_1(C, c)$$

and hence gives rise to an \mathbb{R} -local system \mathbb{V} on C since the transition matrices are constant. In the same way, the trivial line bundle $\tilde{C} \times \mathbb{C} \rightarrow \tilde{C}$ is acted upon by $\pi_1(C, c)$ by

$$(z, \lambda) \mapsto (\gamma(z), (cz + d)^{-1} \lambda)$$

and the inclusion

$$\tilde{C} \times \mathbb{C} \rightarrow \tilde{C} \times \mathbb{C}^2, \quad (z, \lambda) \mapsto (z, \lambda(p(z), 1)^T)$$

is π_1 -equivariant and hence descends to an inclusion of vector bundles $\mathbb{V}^{1,0} \rightarrow \mathbb{V} \otimes_{\mathbb{R}} \mathcal{O}_C$ on C . Since $p(\mathbb{H}) \subseteq \mathbb{H}$, the standard symplectic form on \mathbb{R}^2 with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ furnishes a polarization of this VHS. Moreover, if $\text{im}(\rho) \subset \text{SL}_2(\mathbb{Z})$, then the lattice $\tilde{C} \times \mathbb{Z}^2 \subset \tilde{C} \times \mathbb{R}^2$ is preserved and descends to a \mathbb{Z} -local system $\mathbb{V}_{\mathbb{Z}}$ on C . We put $\mathbb{V}^{0,1} = \mathbb{V} \otimes_{\mathbb{R}} \mathcal{O}_C / \mathbb{V}^{1,0}$. The quotient of $\mathbb{V}^{0,1}$ by the image of $\mathbb{V}_{\mathbb{Z}}$ is then a family of elliptic curves.

2.2. Teichmüller curves. We recall the basic definitions for Teichmüller curves and show that they fit into the above abstract setting with the slight modification that we have to deal with orbifold fundamental groups. Good surveys on this subject are e.g. [McM03], [Möl11] or [HS07].

It is well-known that every Teichmüller curve in \mathcal{M}_g arises as the composition of a Teichmüller embedding $j : \mathbb{H} \rightarrow \mathcal{T}_g$ with the natural projection $\mathcal{T}_g \rightarrow \mathcal{M}_g$, and that a Teichmüller embedding is in turn determined by a pair (X, q) of a compact Riemann surface X with a non-zero quadratic differential q . Using a canonical double covering construction one can confine oneself to $q = \omega^2$, where ω is a holomorphic 1-form. Then the natural atlas on $X \setminus \text{div}(\omega)$ obtained by locally integrating ω has only translations as transition maps, and we call the pair (X, ω) a *translations surface*.

A homeomorphism $f : X \rightarrow X$ is called *affine* if it acts as an affine linear map in the charts of the translation structure. This is the case if and only if its action on $H^1(X, \mathbb{R})$ preserves the subspace spanned by $\text{Re}\omega, \text{Im}\omega$. The group of all orientation-preserving affine homeomorphisms is denoted by $\text{Aff}(X, \omega)$.

Taking the derivative of an affine map induces a group homomorphism

$$D : \text{Aff}(X, \omega) \rightarrow \text{SL}_2(\mathbb{R}),$$

whose image is called the *Veech group* $\text{SL}(X, \omega)$ and whose kernel is the *group of translations* $\text{Trans}(X, \omega)$. The Veech group is a nonuniform discrete subgroup of $\text{SL}_2(\mathbb{R})$ and a lattice if and only if the Teichmüller embedding associated with (X, ω) leads to a Teichmüller curve. In this case, we call the associated surface (X, ω) a *Veech surface* and say that the Teichmüller curve is generated by (X, ω) .

The affine group acts naturally as a subgroup of the mapping class group MCG_g on the Teichmüller disk, respectively as a group of orientation preserving isometries on $\mathbb{H} = \text{SO}(2) \backslash \text{SL}_2(\mathbb{R})$ by the representation D , and the Teichmüller embedding is equivariant for these two actions. This action need not be free, but the kernel $\text{Aut}(X, \omega)$ of affine biholomorphisms of X is always finite. If (X, ω) generates the Teichmüller curve C , the curve $\mathbb{H} / \text{Aff}(X, \omega)$ is the normalization of C and $\text{Aff}(X, \omega)$ is the orbifold fundamental group. In particular, if we view the inclusion $\bar{j} : \mathbb{H} / \text{Aff}(X, \omega) \rightarrow \mathcal{M}_g = \mathcal{T}_g / \text{MCG}_g$ as an inclusion of orbifolds or stacks, we can pull back the universal family over \mathcal{M}_g to obtain a canonical family of curves over a Teichmüller curve. However, to avoid the notion of stacks, we always pass to a suitable finite index subgroup $\Gamma \leq \text{Aff}(X, \omega)$, where a map to a fine moduli space and thus a family $\phi : \mathcal{X} \rightarrow \mathbb{H} / \Gamma$ exists (see [Möl06, 1.4] for details).

2.2.1. Origamis. An *origami*, also called *square-tiled surface* is a translation surface $\mathbf{O} = (X, \omega)$ together with a holomorphic map $p : \mathbf{O} \rightarrow E = \mathbb{C} / \mathbb{Z} \oplus i\mathbb{Z}$ such that p is ramified at most over one point $e \in E$, and such that $\omega = p^*dz$.

Origamis give rise to Veech surfaces, since their Veech group is commensurable with $\text{SL}_2(\mathbb{Z})$. If \mathbf{O} is *primitive*, i. e. p does not factor into $f \circ p'$ where $f : E' \rightarrow E$

is an isogeny between genus 1-surfaces of degree > 1 , then $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup of finite index of $\mathrm{SL}_2(\mathbb{Z})$. The same holds if we consider instead $\mathbf{O}^* = \mathbf{O} \setminus p^{-1}(e)$ and affine maps preserving $p^{-1}(e)$. More on origamis can be found e.g. in [Sch04].

2.2.2. Monodromy representation. The *monodromy representation* of the orbifold fundamental group $\mathrm{Aff}(X, \omega)$ of a Teichmüller curve is the representation

$$\rho : \mathrm{Aff}(X, \omega) \rightarrow \mathrm{Sp}(H^1(X, \mathbb{Z}), i^*), \quad f \mapsto (f^{-1})^*$$

It respects the algebraic intersection pairing i^* on cohomology. One can show that ρ is actually injective and that ρ , restricted to a suitable finite index subgroup where the family $\phi : \mathcal{X} \rightarrow \mathbb{H}/\Gamma$ exists, is the monodromy representation associated with $R^1\phi_*\mathbb{Z}$ (see [Bau09] for the proof of both statements).

In the case of a Teichmüller curve, the equivariance carries over to the possibly non-free action of $\mathrm{Aff}(X, \omega)$ on \mathbb{H} and on \mathbb{H}_k via its monodromy representation. This is easily seen as follows. The Teichmüller embedding $j : \mathbb{H} \rightarrow \mathcal{T}_g = \mathcal{T}(X)$ associated with (X, ω) is equivariant with respect to the action of $f \in \mathrm{Aff}(X, \omega)$ by $D(f)$ on \mathbb{H} and by its action as element of the mapping class group, that sends the marked Riemann surface (X_τ, m_τ) to $(X_\tau, m_\tau \circ f^{-1})$. The natural map $t : \mathcal{T}(X) \rightarrow \mathbb{H}_g$ is in turn equivariant with respect to the Torelli morphism $\mathrm{MCG}_g \rightarrow \mathrm{Sp}(2g, \mathbb{Z})$, $f \mapsto (f^{-1})^*$. The period map $p_{\phi_{\mathrm{univ}}}$ of the pullback family $\phi_{\mathrm{univ}} : \mathcal{X} = \mathcal{X}_{\mathrm{univ}} \times_j \mathbb{H} \rightarrow \mathbb{H}$ of the universal family of curves $\mathcal{X}_{\mathrm{univ}} \rightarrow \mathcal{T}_g$ is now given as the composition of $t \circ j$.

If Γ is a finite-index subgroup preserving a subspace W of $H^1(X, \mathbb{R})$, then the associated representation will induce a sub-local system \mathbb{W} of $R^1\phi_*\mathbb{R}$ on some \mathbb{H}/Γ' for a suitable finite index subgroup $\Gamma' \leq \Gamma$. Applying Deligne's semisimplicity result, we find that \mathbb{W} carries a VHS, and $R^1\phi_*\mathbb{R} = \mathbb{W} \oplus \tilde{\mathbb{W}}$ where $\tilde{\mathbb{W}}$ is the complement of \mathbb{W} . Therefore, we can find a trivialization of the pullback local system on \mathbb{H} , i.e. a basis of $H_1(X, \mathbb{R})$ such that with respect to this basis, the period map $p_{\phi_{\mathrm{univ}}}$ is given as

$$z \mapsto \begin{pmatrix} Z_1(z) & 0 \\ 0 & Z_2(z) \end{pmatrix} \in \mathbb{H}_g,$$

where Z_1 and Z_2 are square matrices of dimensions $\mathrm{rk} \mathbb{W}^{1,0}$ and $\mathrm{rk} \tilde{\mathbb{W}}^{1,0}$. This map is equivariant for all $\gamma \in \mathrm{Aff}(X, \omega)$ such that $\rho(\gamma)$ respects the decomposition $W \oplus \tilde{W}$. In particular, the period map

$$p_W : \mathbb{H} \rightarrow \mathbb{H}_{\mathrm{rk} W}, \quad z \mapsto Z_1(z)$$

associated with the VHS \mathbb{W} is Γ -equivariant (and not just Γ' -equivariant).

2.2.3. The VHS of the family of curves over a Teichmüller curve. Using Deligne's result, Möller characterizes the VHS on a Teichmüller curve [Möl06] generated by a translation surface (X, ω) . After passing to a finite cover, the VHS on a Teichmüller curve always admits a uniformizing direct factor \mathbb{L} in its VHS, defined over the trace field of $\mathrm{SL}(X, \omega)$, whose local system is given by the Fuchsian representation D of $\mathrm{Aff}(X, \omega)$. Conversely, he shows that if a family of curves $\phi : \mathcal{X} \rightarrow C$ over a curve C has a uniformizing direct summand \mathbb{L} in its \mathbb{R} -VHS $R^1\phi_*\mathbb{R}$, then C is a finite cover of a Teichmüller curve.

2.3. Lyapunov exponents. Lyapunov exponents are characteristic numbers associated with certain dynamical systems. In our case of a \mathbb{R} -VHS \mathbb{V} on a hyperbolic curve $C = \mathbb{H}/\Gamma$, they measure the logarithmic growth rate of the Hodge norm of a vector in \mathbb{V}_x when being dragged along a generic (w.r.t. the Haar measure) geodesic on \mathbb{H}/Γ under parallel transport.

For an \mathbb{R} -VHS of rank $2k$, the Lyapunov spectrum consists of $2k$ exponents, counted with multiplicity that group symmetrically around 0

$$\lambda_1 \geq \cdots \geq \lambda_k \geq 0 \geq \lambda_{k+1} = -\lambda_k \geq \cdots \geq \lambda_{2k} = -\lambda_1.$$

One usually normalizes the curvature in order that $\lambda_1 = 1$ ($K = -4$ in the case of hyperbolic curves). In the case of a Teichmüller curve, we further have $\lambda_1 = 1 > \lambda_2$. In general, virtually all knowledge about individual exponents stems from using variants of a formula for the sum over the first half of the spectrum which we refer to as the *non-negative Lyapunov spectrum* in the following. This formula is originally due to Kontsevich and Zorich [KZ97], and was rigorously proved in [BM10], [EKZ10] or [For02]. A variant of it can be stated as follows.

Theorem 2.1. *Let \mathbb{V} be an \mathbb{R} -VHS of weight 1 and rank $2k$ on a (possibly non-compact) curve $C = \mathbb{H}/\Gamma$. Then the non-negative Lyapunov exponents $\lambda_1, \dots, \lambda_k$ of \mathbb{V} satisfy*

$$(2.2) \quad \lambda_1 + \cdots + \lambda_k = \frac{2 \deg(\mathcal{V}^{1,0})}{2g(\overline{C}) - 2 + s}$$

where \overline{C} is the completion of C , $s = |\overline{C} \setminus C|$, and $\mathcal{V}^{1,0}$ is the Deligne extension of $\mathbb{V}^{1,0}$ to \overline{C} .

A generalization of this formula to higher dimensional ball quotients also exists [KM12], as well as an explicit formula for the sum of Lyapunov exponents of the relative cohomology in case the Teichmüller curve is generated by an origami [EKZ10].

Using Theorem 2.1, individual Lyapunov exponents have been computed e.g. for families of cyclic and abelian coverings of \mathbb{P}^1 ramified over 4 points ([EKZ11], [Wri12a]), in genus two [Bai07] and for all known primitive Teichmüller curves in higher genus [BM10].

We recall two important properties of the Lyapunov spectrum. First, it remains unchanged if we pass to a finite index subgroup Γ' and consider the Lyapunov spectrum of the pullback VHS on \mathbb{H}/Γ' (see e.g. [KM12, Proposition 5.6]). Secondly, if the VHS splits up as a direct sum, then its Lyapunov spectrum is the union of the spectra of its pieces.

3. LYAPUNOV EXPONENTS OF RANK 2-VHS

In this section, we derive the main theorem from Theorem 2.1.

Proposition 3.1. *Let $\rho : \Gamma \rightarrow \mathrm{SL}_2(\mathbb{R})$ be a group homomorphism such that Γ and $\Delta = \rho(\Gamma)$ are cofinite, torsionfree Fuchsian groups, and let $p : \mathbb{H} \rightarrow \mathbb{H}$ be a non-constant ρ -equivariant holomorphic map. Let $\overline{p} : \mathbb{H}/\Gamma \rightarrow \mathbb{H}/\Delta$ be the map induced by p , and let \mathbb{V} be the pullback by \overline{p} of the universal rank-2 \mathbb{R} -VHS on \mathbb{H}/Δ . Then the non-negative Lyapunov exponent of \mathbb{V} is given by*

$$(3.1) \quad \lambda = \frac{\deg(\overline{p}) \mathrm{vol}(\mathbb{H}/\Delta)}{\mathrm{vol}(\mathbb{H}/\Gamma)}.$$

Proof. By Theorem 2.1, the Lyapunov exponent is given by

$$\lambda = \frac{2 \deg(\mathcal{V}^{1,0})}{\deg(\omega_{\overline{B}})},$$

where \overline{B} is the completion of \mathbb{H}/Γ and where $\mathcal{V}^{1,0}$ is the Deligne extension to \overline{B} of the $(1,0)$ -part of \mathbb{V} . Further,

$$\deg(\omega_{\overline{B}}) = -\chi(\overline{B}) = \frac{1}{2\pi} 4 \operatorname{vol}(\mathbb{H}/\Gamma),$$

by the Gauß-Bonnet formula (where we take the curvature on \mathbb{H} to be normalized to -4). Let \overline{C} be the completion of \mathbb{H}/Δ , and let \mathcal{U} be the universal VHS on \mathbb{H}/Δ , whose Deligne extension of the $(1,0)$ -part we denote by $\mathcal{U}^{1,0}$. By universality and the Gauß-Bonnet formula, we have

$$2 \deg(\mathcal{U}^{1,0}) = \deg(\omega_{\overline{C}}) = \frac{1}{2\pi} 4 \operatorname{vol}(\mathbb{H}/\Delta),$$

and since $\overline{p}^* \mathcal{U}^{1,0} = \mathcal{V}^{1,0}$, the claim follows. \square

We remark that Proposition 3.1 is also readily deduced from a reformulation of the Kontsevich-Zorich formula by Wright [Wri12a, Theorem 1.2].

For our applications, we need to allow groups that contain torsion elements or whose action on \mathbb{H} has a (usually finite) kernel. In this situation there might not be a VHS on the quotient, but only on an appropriate finite cover. (Note that by a theorem of Selberg, any finitely generated subgroup of a matrix group always has a torsionfree subgroup of finite index.) However, we still can compute the right-hand side of (3.1). The next lemma shows that this quantity is independent under passing to a finite index subgroup.

Lemma 3.2. *Let Γ be a group acting cofinitely and holomorphically on \mathbb{H} . Let $\rho : \Gamma \rightarrow \operatorname{SL}_2(\mathbb{R})$ be a group homomorphism such that $\Delta = \rho(\Gamma)$ is a cofinite Fuchsian group, and let $p : \mathbb{H} \rightarrow \mathbb{H}$ be a non-constant ρ -equivariant holomorphic map. Let $\Gamma' \leq \Gamma$ be a finite index subgroup. Then $\Delta' = \rho(\Gamma')$ has finite index and*

$$\frac{\deg(\overline{p}) \operatorname{vol}(\mathbb{H}/\Delta)}{\operatorname{vol}(\mathbb{H}/\Gamma)} = \frac{\deg(\overline{p}') \operatorname{vol}(\mathbb{H}/\Delta')}{\operatorname{vol}(\mathbb{H}/\Gamma')},$$

where $\overline{p} : \mathbb{H}/\Gamma \rightarrow \mathbb{H}/\Delta$ and $\overline{p}' : \mathbb{H}/\Gamma' \rightarrow \mathbb{H}/\Delta'$ are the maps induced by p .

Proof. We have $(\Delta : \Delta') = (\Gamma : \rho^{-1}(\Delta')) \leq (\Gamma : \Gamma')$. The second claim follows by comparing the degrees of maps in the commutative diagram

$$\begin{array}{ccc} \mathbb{H}/\Gamma' & \xrightarrow{\overline{p}'} & \mathbb{H}/\Delta' \\ \downarrow & & \downarrow \\ \mathbb{H}/\Gamma & \xrightarrow{\overline{p}} & \mathbb{H}/\Delta \end{array}$$

\square

3.1. Computing the degree of \overline{p} . In this section, we show that in the presence of cusps, the quantities on the right hand side of (3.1) are explicitly computable only from the group homomorphism ρ .

Throughout, let $\rho : \Gamma \rightarrow \Delta$ be a homomorphism between non-cocompact, cofinite Fuchsian groups, let $p : \mathbb{H} \rightarrow \mathbb{H}$ be a ρ -equivariant non-constant holomorphic map, and let $\overline{p} : \mathbb{H}/\Gamma \rightarrow \mathbb{H}/\Delta$ be the map induced by p . Denote the extension $\overline{p} : \overline{B} \rightarrow \overline{C}$ to the completions \overline{B} of \mathbb{H}/Γ and \overline{C} of \mathbb{H}/Δ by the same letter.

In the following, a cusp will, depending on the context, be a point in $\partial\mathbb{H}$, stabilized by a parabolic in Γ or its equivalence class under the action of Γ , respectively the point in the completion of \overline{B} corresponding to this class.

Lemma 3.3. *Let $\Gamma_0 \leq \Gamma$, respectively $\Delta_0 \leq \Delta$ be maximal parabolic subgroups associated with cusps $b \in \overline{B}$, respectively $c \in \overline{C}$. Let γ be a generator of Γ_0 such that $\rho(\gamma)$ is parabolic and lies in Δ_0 . Then*

- a) \overline{p} maps b to c .
- b) The ramification index $e(\overline{p}, b)$ of \overline{p} at b is $(\Delta_0 : \rho(\Gamma_0))$.
- c) We have $\deg(\overline{p}) = \sum_{b \in \overline{p}^{-1}(c)} e(\overline{p}, b)$.

Proof. Let s , respectively $t \in \mathbb{R} \cup \{\infty\}$ be the fixed point of Γ_0 , respectively Δ_0 . Without loss of generality, we may assume $s = t = \infty$, and that Γ_0 respectively Δ_0 is generated by $(z \mapsto z + 1)$. The canonical projections $u_\Gamma : \mathbb{H} \rightarrow \mathbb{H}/\Gamma$ respectively $u_\Delta : \mathbb{H} \rightarrow \mathbb{H}/\Delta$ factor over $\mathbb{H} \rightarrow \mathbb{H}/\Gamma_0$ respectively $\mathbb{H} \rightarrow \mathbb{H}/\Delta_0$, and both \mathbb{H}/Γ_0 and \mathbb{H}/Δ_0 are isomorphic to \mathbb{D}^* via the map induced by $z \mapsto \exp(2\pi iz)$. Under this isomorphism, the image of s , respectively t is identified with $0 \in \mathbb{D}$. Being equivariant, the map p descends to $p_0 : \mathbb{H}/\Gamma_0 \cong \mathbb{D}^* \rightarrow \mathbb{D}^* \cong \mathbb{H}/\Delta_0$.

To prove a), it suffices to show that for a sequence in \mathbb{D}^* converging to 0, the image under p_0 converges to 0. Define $a_n = in$ and let $b_n = \exp(2\pi ia_n)$ in \mathbb{D}^* ; we have $b_n \rightarrow 0$. By the Schwarz lemma, p does not increase hyperbolic distances, thus

$$d_{\text{hyp}}(a_n, a_n + 1) \geq d_{\text{hyp}}(p(a_n), p(a_n) + \lambda),$$

where $z \mapsto z + \lambda$, ($\lambda \in \mathbb{Z} \setminus \{0\}$) generates $\rho(\Gamma_0)$. Since $d_{\text{hyp}}(a_n, a_n + 1) \rightarrow 0$ as $n \rightarrow \infty$, we also have $d_{\text{hyp}}(p(a_n), p(a_n) + \lambda) \rightarrow 0$, whence $\text{Im}(p(a_n)) \rightarrow \infty$, which means that $p_0(b_n) \rightarrow 0$.

b) A basis of punctured neighborhoods of $b \in \overline{B}$ is given by the images of horoballs $U_R = \{z \in \mathbb{H} \mid \text{Im}(z) > R\}$ under the projection modulo Γ . If we choose R big enough, then we can ensure that U_R is stabilized only by elements of Γ_0 , whence the quotient U_R/Γ_0 embeds into \mathbb{H}/Γ , and gives rise to a chart $U_R/\Gamma_0 \rightarrow \mathbb{D}^*$. In the same way, we can obtain a chart $U_{R'}/\Delta_0 \rightarrow \mathbb{D}^*$ such that in these charts, \overline{p} takes the form $z \mapsto z^k$ with k being the ramification index. Thus the induced map \overline{p}_* on fundamental groups maps a generator of $\pi_1(U_R/\Gamma_0) \cong \Gamma_0$ to the k -th power of a generator of $\pi_1(U_{R'}/\Delta_0) \cong \Delta_0$. This group homomorphism $\Gamma_0 \rightarrow \Delta_0$ must be equal to ρ , since for both p is equivariant. It follows that $k = (\Delta_0 : \rho(\Gamma_0))$. \square

Note that the degree of \overline{p} can be 1 without p being an isomorphism. However, this can happen only when the Fuchsian groups contain torsion elements.

Proof of Theorem 1.1. The Lyapunov spectrum does not change, if we pass to a finite index subgroup Γ' of Γ . Thus if $\rho_V(\Gamma)$ is finite, then $\rho_V(\Gamma')$ will be trivial for the finite index subgroup $\Gamma' = \text{Ker}(\rho_V)$, and therefore $\lambda = 0$.

We are left with the case when $\rho_V(\Gamma)$ is infinite. By Deligne's semisimplicity theorem, the local system \mathbb{V} associated with V carries a VHS. We let p_V be its period map. p_V cannot be constant, for otherwise every $g \in \rho_V(\Gamma)$ would stabilize $p(z) \equiv \text{const} \in \mathbb{H}$, but this stabilizer is finite since $\rho_V(\Gamma)$ is discrete. Thus we obtain a non-constant holomorphic map $p : \mathbb{H} \rightarrow \mathbb{H}$ that descends to $\overline{p} : \mathbb{H}/\Gamma \rightarrow \mathbb{H}/\rho_V(\Gamma)$. On the left-hand side, we have a Riemann surface of finite type. We claim that \overline{p} can be extended continuously to the compactification \overline{B} of \mathbb{H}/Γ , respectively the possibly only partial compactification \overline{C} of $\mathbb{H}/\rho_V(\Gamma)$, where \overline{B} and \overline{C} are obtained

by adjoining all cusps. From this we conclude that \overline{C} is compact and thus $\mathbb{H}/\rho_V(\Gamma)$ has finite volume.

To prove the claim, let $b \in \partial\mathbb{H}$ be a cusp of Γ and let γ be a generator of its stabilizer. By the Schwarz lemma, it follows that

$$d_{\mathbb{H}}(z, \gamma(z)) \geq d_{\mathbb{H}}(p(z), p(\gamma(z))) \geq \ell(\rho(\gamma))$$

where $\ell(g) = \inf_{z \in \mathbb{H}} d_{\mathbb{H}}(z, g(z))$ is the translation length. Since the left-hand side goes to 0 as z approaches the cusp, $\ell(\rho(\gamma)) = 0$, whence $\rho(\gamma)$ is either parabolic or elliptic. In the first case, the proof of Lemma 3.3 a) shows that \overline{p} is locally given as a holomorphic map $\mathbb{D}^* \rightarrow \mathbb{D}^*$, which has a canonical extension to $\mathbb{D} \rightarrow \mathbb{D}$. This is true also for the second case with the difference that $\overline{p}(b)$ is now a point in $\mathbb{H}/\rho_V(\Gamma)$.

The statement of Theorem 1.1 now follows from Proposition 3.1 together with Lemma 3.3. \square

4. APPLICATIONS

In this section, we describe how to algorithmically obtain the monodromy representation in the case of origamis in terms of the action of generators of the affine group. Then we exhibit two principles to split up this representation into subrepresentations. As an application, we present two examples where a splitting of the monodromy representation of a Teichmüller curve into rank 2-subrepresentations is found. We then use the technique from the previous section to determine the Lyapunov spectrum.

4.1. Algorithmic approach. Given an origami $p : \mathbf{O} \rightarrow E$, we outline an algorithm for obtaining the monodromy representation of $\text{Aff}(\mathbf{O})$ in terms of its generators. It has been realized mainly by Myriam Finster, building on work of Gabriela Weitze-Schmithüsen, Karsten Kremer and others.

To fix notations, let E^* be E minus the ramification point e of p , and let $\mathbf{O}^* = \mathbf{O} \setminus p^{-1}(e)$. Then $p : \mathbf{O}^* \rightarrow E^*$ is a topological covering. We fix an isomorphism $\pi_1(E^*) \cong F_2$ by choosing the basis x, y of $\pi_1(E^*)$ represented by a horizontal and a vertical path in E^* . The preimage of $x \cup y$ under p is a 4-valent graph $\mathcal{G}(\mathbf{O}^*)$ homotopy-equivalent to \mathbf{O}^* . Moreover, $\pi_1(\mathbf{O}^*)$ injects into $\pi_1(E^*)$; let p_*g be this injection and let its image be denoted by $H = H(\mathbf{O}^*)$.

We make use of a proposition, which is already implicit in [Sch04]. Let $c : F_2 \rightarrow \text{Aut}^+(F_2)$ denote the canonical inclusion of the inner automorphisms of F_2 into its orientation-preserving automorphisms and let $\beta : \text{Aut}^+(F_2) \rightarrow \text{SL}_2(\mathbb{Z}) \cong \text{Out}^+(F_2)$ denote the canonical projection.

Proposition 4.1. *Let $p : \mathbf{O} \rightarrow E$ be an origami, and let $H = H(\mathbf{O}^*)$. There is a commutative diagram with exact rows*

$$\begin{array}{ccccccc} 1 & \longrightarrow & N(H)/H & \xrightarrow{c} & \text{Stab}^+(H)/c(H) & \xrightarrow{\beta} & \text{SL}(\mathbf{O}^*) \longrightarrow 1 \\ & & \uparrow \cong & & \uparrow \cong & & \parallel \\ 1 & \longrightarrow & \text{Trans}(\mathbf{O}^*) & \longrightarrow & \text{Aff}(\mathbf{O}^*) & \xrightarrow{D} & \text{SL}(\mathbf{O}^*) \longrightarrow 1 \end{array}$$

where $\text{Stab}^+(H)$ is the subset of $f \in \text{Aut}^+(F_2)$ such that $f(H) = H$, and $N(H)$ is the normalizer of H in F_2 .

Moreover, the injection p_* is equivariant for the actions by outer automorphisms of $f \in \text{Aff}(\mathbf{O}^*)$ on $\pi_1(\mathbf{O}^*)$ and of $\psi(f) \cdot c(H) \in \text{Stab}^+(H)/c(H)$ on H .

Note that in general $\text{Aff}(\mathbf{O}) \supsetneq \text{Aff}(\mathbf{O}^*)$ if \mathbf{O} is not a primitive origami. Also $\text{Trans}(\mathbf{O}^*) = \text{Trans}(\mathbf{O})$ only holds if $g(\mathbf{O}) > 2$.

Proof. Let $u : \tilde{X} \rightarrow \mathbf{O}^*$ denote a fixed universal covering, and endow it with the translation structure obtained by pullback. Then $p \circ u : \tilde{X} \rightarrow E^*$ is a universal covering of E^* . Let $\text{Gal}(\tilde{X}/E^*)$ denote the deck transformations of $p \circ u$. By [Sch04], there is a commutative diagram with exact rows

$$(4.1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & F_2 & \xrightarrow{c} & \text{Aut}^+(F_2) & \xrightarrow{\beta} & \text{SL}_2(\mathbb{Z}) \longrightarrow 1 \\ & & \uparrow \cong & & \uparrow \cong & & \parallel \\ 1 & \longrightarrow & \text{Gal}(\tilde{X}/E^*) & \longrightarrow & \text{Aff}(\tilde{X}) & \xrightarrow{D} & \text{SL}_2(\mathbb{Z}) \longrightarrow 1 \end{array}$$

where the isomorphism $\text{Aff}(\tilde{X}) \rightarrow \text{Aut}^+(F_2)$ stems from the fact that each affine $f : \tilde{X} \rightarrow \tilde{X}$ descends to E^* and induces an orientation preserving automorphism of F_2 . Define $\text{Aff}_u(\tilde{X})$ to be the subgroup of affine automorphisms descending to \mathbf{O} via u , and let $\text{Trans}_u(\tilde{X}) = \text{Aff}_u(\tilde{X}) \cap \text{Gal}(\tilde{X}/E^*)$. We claim that we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 1 & \longrightarrow & N(H) & \xrightarrow{c} & \text{Stab}^+(H) & \xrightarrow{\beta} & \text{SL}(\mathbf{O}^*) \longrightarrow 1 \\ & & \uparrow \cong & & \uparrow \cong & & \parallel \\ 1 & \longrightarrow & \text{Trans}_u(\tilde{X}) & \longrightarrow & \text{Aff}_u(\tilde{X}) & \xrightarrow{D} & \text{SL}(\mathbf{O}^*) \longrightarrow 1 \end{array}$$

The bottom row is exact by the definition of $\text{Trans}_u(\tilde{X})$ and the fact that the canonical projection $\text{Aff}_u(\tilde{X}) \rightarrow \text{Aff}(\mathbf{O}^*)$ is surjective. Again by [Sch04], the image of $\text{Aff}_u(\tilde{X})$ in $\text{Aut}^+(F_2)$ is precisely $\text{Stab}^+(H)$ and the image of $\text{Stab}^+(H)$ under β is $\text{SL}(\mathbf{O}^*)$. Finally, $c(F_2) \cap \text{Stab}^+(H) = c(N(H))$.

The first claim of the proposition now follows from the fact that the kernel of the canonical projection $\text{Aff}_u(\tilde{X}) \rightarrow \text{Aff}(\mathbf{O}^*)$ is precisely $\text{Gal}(\tilde{X}/\mathbf{O}^*) \cong H$.

As to the second claim, the description of the isomorphism $\text{Aff}(\tilde{X}) \rightarrow \text{Aut}^+(F_2)$ implies that $\psi(f)$ is the class of \bar{f}_* , where \bar{f} is the map induced by f on E^* . Thus for every path $\gamma \in \pi_1(\mathbf{O})$, $\psi(f)(p_*\gamma) = \bar{f}_*p_*\gamma = p_*f_*\gamma$. Since p_* is an isomorphism onto its image, it maps the conjugacy class of $f_*\gamma$ in $\pi_1(\mathbf{O})$ to the conjugacy class of $\psi(f)(p_*\gamma)$, which proves the claim. \square

The input of our algorithm is an origami $p : \mathbf{O} \rightarrow E$ be an origami of degree $\deg p = d$, given as graph $\mathcal{G}(\mathbf{O}^*)$.

Step 1: Construct a basis of $\pi_1(\mathbf{O}^*)$. Choose a maximal spanning tree T in $\mathcal{G}(\mathbf{O}^*)$.

The edges t_1, \dots, t_{d+1} not in T represent a basis of $\pi_1(\mathbf{O}^*)$. Mapping this basis to $H \leq F_2$, we obtain a free system of generators u_1, \dots, u_{d+1} for H .

Step 2: Compute a system of generators $\gamma_1, \dots, \gamma_r$ of $\text{Stab}^+(H)$ (see [Fin08]).

Step 3: Lift the action of γ_i on the generators of H to an action on t_1, \dots, t_{d+1} .

Let $w_{ij} = \gamma_i(u_j)$; this is a word in x, y which can be decomposed as a word in the generators of $\pi_1(\mathbf{O}^*)$ by writing down all non-tree edges crossed on the path in $\mathcal{G}(\mathbf{O}^*)$ determined by w_{ij} .

Step 4: Find an extended symplectic basis of $\pi_1(X^*)$ by surface normalization as in [Sti80].

Step 5: For each generator γ_i of $\text{Stab}^+(H)$, project its action on the generators of $\pi_1(\mathbf{O}^*)$ to $\text{GL}(H_1(\mathbf{O}^*, \mathbb{Z}))$. Then make a base change to the extended

symplectic basis found in Step 4. Discard the basis elements representing loops around the punctures to obtain the action of $\text{Aff}(\mathbf{O}^*)$ on $H_1(\mathbf{O}, \mathbb{Z})$.

Proposition 4.1 implies the correctness of the above algorithm.

4.2. Splitting principles. We describe two principles for finding subrepresentations of a monodromy representation.

Given two Veech surfaces (X, ω) , (Y, ν) , we call a non-constant holomorphic map $f : X \rightarrow Y$ a *Veech covering* if $f^*\nu = \omega$ and if the Veech group of Y minus the ramification points of f is a lattice. Note that this happens if and only if all branch points are periodic points, i.e. have finite $\text{Aff}(Y, \nu)$ -orbits.

A Veech covering $p : (X, \omega) \rightarrow (Y, \nu)$ between Veech surfaces induces a subrepresentation as follows. By [GJ00, Theorem 4.8] the elements of $\text{Aff}(X, \omega)$ that descend via p to Y form a finite-index subgroup $\text{Aff}(X, \omega)_p$ of $\text{Aff}(X, \omega)$. Let

$$\varphi_p : \text{Aff}(X, \omega)_p \rightarrow \text{Aff}(Y, \nu)$$

be the group homomorphism that maps $f \in \text{Aff}(X, \omega)_p$ to $\bar{f} \in \text{Aff}(Y, \nu)$ such that $p \circ f = \bar{f} \circ p$. The image of φ_p is the finite-index subgroup $\text{Aff}(Y, \nu)^p$ of $\text{Aff}(Y, \nu)$ of affine diffeomorphisms, that lift to (X, ω) .

Proposition 4.2. *Let $p : (X, \omega) \rightarrow (Y, \nu)$ be a Veech covering between Veech surfaces and let $\rho : \text{Aff}(X, \omega) \rightarrow \text{Sp}(H^1(X, \mathbb{Z}))$ be the monodromy representation of (X, ω) . Then the image U of $H^1(Y, \mathbb{Z})$ under*

$$p^* : H^1(Y, \mathbb{Z}) \rightarrow H^1(X, \mathbb{Z})$$

is an $\text{Aff}(X, \omega)_p$ -invariant symplectic subspace of $H^1(X, \mathbb{Z})$ polarized by $\deg(p) \cdot Q_X$.

The map p^ is equivariant for the action of $\text{Aff}(X, \omega)_p$ on U and $\text{Aff}(Y, \nu)^p$ on $H^1(Y, \mathbb{Z})$.*

Proof. Let $f \in \text{Aff}(X, \omega)_p$ and $\bar{f} \in \text{Aff}(Y, \nu)$ such that $p \circ f = \bar{f} \circ p$. Then for every $c \in H^1(Y, \mathbb{Z})$

$$(f^{-1})^*(p^*(c)) = (p \circ f^{-1})^*(c) = (\bar{f}^{-1} \circ p)^*(c) = p^*((\bar{f}^{-1})^*(c)) ,$$

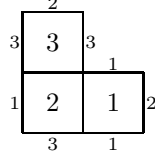
proving $(f^{-1})^*(\text{Im}(p^*)) \subset \text{Im}(p^*)$. The computation also shows that p^* is equivariant. Finally, p^* is a symplectic map and $Q_X(p^*c_1, p^*c_2) = \deg p \cdot Q_Y(c_1, c_2)$. \square

We note that the uniformizing subrepresentation of an origami is induced by the Veech covering $p : \mathbf{O} \rightarrow E$.

Secondly, the group $\text{Aut}(X, \omega)$ of affine biholomorphisms acts on $H^1(X, \mathbb{R})$ and $H^1(X, \mathbb{C})$ and we can use representation theory of finite groups to decompose these vector spaces into a direct sum $\mathbb{K}[\text{Aut}(X, \omega)]$ -modules (with $\mathbb{K} = \mathbb{R}$ or \mathbb{C}). This technique has been successfully applied in [MYZ12].

Proposition 4.3. *Let (X, ω) be a Veech surface and let $G \leq \text{Aut}(X, \omega)$. The action of $\text{Aff}(X, \omega)$ on $H^1(X, \mathbb{K})$, restricted to the normalizer $N(G)$ of G in $\text{Aff}(X, \omega)$, permutes the isotypic components of the decomposition of $H^1(X, \mathbb{K})$ into G -modules and there is a finite index subgroup $\Gamma \leq \text{Aff}(X, \omega)$ such that every isotypic component is Γ -invariant.*

Proof. As $\text{Aut}(X, \omega)$ is normal in $\text{Aff}(X, \omega)$, the normalizer $N(G)$ of G in $\text{Aff}(X, \omega)$ has finite index in $\text{Aff}(X, \omega)$. For all $g \in G$, and $f \in N(G)$, there exists $\tilde{g} \in G$,


 FIGURE 1. *The origami $\mathbf{L}_{2,2}$*

such that $gf = f\tilde{g}$. Therefore for all irreducible $\mathbb{K}[G]$ -submodules V of $H^1(X, \mathbb{K})$, we have

$$(g^*)^{-1} \circ (f^*)^{-1}(V) = ((gf)^*)^{-1}(V) = ((f\tilde{g})^*)^{-1}(V) = (f^*)^{-1}(V),$$

which shows that $(f^{-1})^*(V)$ is another irreducible $\mathbb{K}[G]$ -module inside $H^1(X, \mathbb{K})$. Hence every $f \in N(G)$ induces a permutation of the isotypic components of the representation of G . Thus there is a finite index subgroup $\Gamma \leq N(G)$ that leaves every isotypic component invariant. \square

Remark 4.4. In both cases, the subrepresentations carry a VHS. This follows directly from Deligne's semisimplicity theorem.

There can be invariant subspaces not directly related to these two constructions due to hidden symmetries of the Jacobian (e.g. endomorphisms of Hecke type as discussed in [Ell01]).

4.3. Examples. The examples discussed in the following are both origamis and stem from intermediate covers of the characteristic origami $\widetilde{\mathbf{St}}_3$ discussed in [Her06]. We remain rather brief here; a complete discussion including all matrix computations is found in [Kap11].

We note that in our examples the individual Lyapunov exponents can also be obtained from the formula for their sum, combined with knowledge on intermediate coverings.

In the following, denote $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

First example. Let $\mathbf{L}_{2,2}$ be the origami given by

$$r = (1 \ 2)(3), \quad u = (1 \ 3)(2),$$

where the permutation r (u) is the monodromy the horizontal (vertical) generator of $\pi_1(E^*)$. It is the smallest origami of genus 2. Its affine group $\mathrm{SL}(\mathbf{L}_{2,2})$ is isomorphic to the index 3 subgroup Γ_Θ of $\mathrm{SL}_2(\mathbb{Z})$ generated by S, T^2 . It follows that $\mathrm{SL}(\mathbf{L}_{2,2})$ is isomorphic to the orientation preserving subgroup of the $\Delta(2, \infty, \infty)$ -triangle group of the hyperbolic triangle with vertices $(i, \infty, 1)$. In particular, the stabilizer of 1 is generated by T^2S .

The action of $\mathrm{Aff}(\mathbf{L}_{2,2})$ on $H^1(\mathbf{L}_{2,2}, \mathbb{Z})$ splits over \mathbb{Q} into the uniformizing representation and a second representation $\rho_{\mathbf{L}_{2,2},2} : \mathrm{Aff}(\mathbf{L}_{2,2}) \rightarrow \mathrm{SL}_2(\mathbb{Z})$ given in an appropriate basis by

$$T^2 \mapsto T, \quad S \mapsto S^{-1}.$$

Proposition 4.5. *The non-negative Lyapunov exponent associated with $\rho_{\mathbf{L}_{2,2},2}$ is $1/3$.*

Proof. Let $p : \mathbb{H} \rightarrow \mathbb{H}$ denote the period map of the VHS associated with $\rho_{\mathbf{L}_{2,2,2}}$. Since $T^2 \mapsto T$, while $T^2 S \mapsto TS$, an element of order 3, the preimage of the cusp $i\infty$ of $\mathrm{SL}_2(\mathbb{Z})$ under p is only the cusp $i\infty$. By Lemma 3.3, $\deg \bar{p} = 1$. By Proposition 3.1 and Lemma 3.2,

$$\lambda = \frac{\mathrm{vol}(\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}))}{\mathrm{vol}(\mathbb{H}/\Gamma_\Theta)} = 1/3.$$

□

Note that this matches Bainbridge's result on Lyapunov exponents of invariant measures on $\Omega\mathcal{M}_2$ [Bai07].

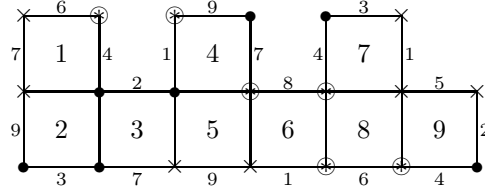


FIGURE 2. The origami \mathbf{M}

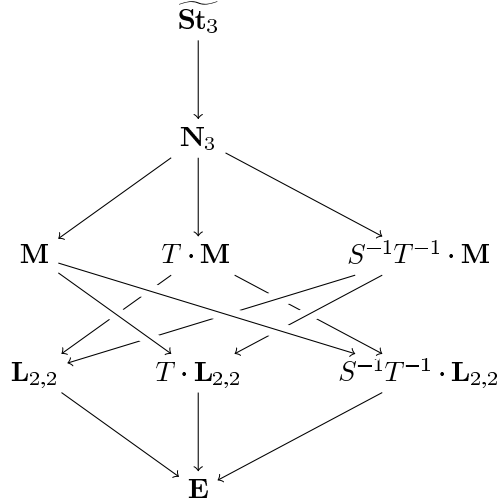


FIGURE 3. Intermediate covers of $\widetilde{\mathbf{St}_3}$. $\mathrm{SL}_2(\mathbb{Z})$ -orbits of $\mathbf{L}_{2,2}$, \mathbf{M} and a third origami \mathbf{N}_3 with 27 squares and Veech group $\mathrm{SL}_2(\mathbb{Z})$. Arrows indicate Veech covering maps

Second example. The second example is the origami \mathbf{M} (see Figure 2) given by

$$r = (1, 4, 7)(2, 3, 5, 6, 8, 9) \quad \text{and} \quad u = (1, 6, 8, 7, 3, 2)(4, 9, 5),$$

and belongs to $\Omega\mathcal{M}_4(2, 2, 2)^{\text{odd}}$. Its affine group is also equal to Γ_Θ . The monodromy representation $\rho_{\mathbf{M}} : \text{Aff}(\mathbf{M}) \rightarrow H^1(\mathbf{M}, \mathbb{Z})$ restricted to $\Gamma(2) \leq \text{SL}(\mathbf{M})$ splits over \mathbb{Q} into four symplectic subrepresentations, each of rank 2. Apart from the uniformizing representation ρ_1 , there are two representations ρ_2, ρ_3 that are induced from coverings to genus 2-origamis. Figure 3 shows the poset of intermediate covers of \mathbf{St}_3 .

Let us denote $\rho_{\mathbf{M},4}$ the representation complementary to $\rho_{\mathbf{M},1} \oplus \rho_{\mathbf{M},2} \oplus \rho_{\mathbf{M},3}$. It already splits off over $\text{SL}(\mathbf{M})$, and is given by

$$\rho_{\mathbf{M},4}(T^2) = T^{-1}S = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho_{\mathbf{M},4}(S) = S^{-1}.$$

Proposition 4.6. *The non-negative Lyapunov spectrum of \mathbf{M} is*

$$1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}.$$

Proof. By Proposition 4.2, each of the coverings to the genus 2-origamis induces a rank 4-subspace invariant under some finite index subgroup Γ (we can take $\Gamma = \Gamma(2)$) of $\text{Aff}(\mathbf{M})$. The pullback of the uniformizing representation is the uniformizing representation upstairs. Furthermore a computation shows that the pullbacks of the non-uniformizing representations are distinct, whence two rank 2-representations $\rho_{\mathbf{M},2}$ and $\rho_{\mathbf{M},3}$. Both being pulled back from origamis in $\Omega\mathcal{M}_2(2)$, they have Lyapunov exponent $1/3$. As in the first example, the third Lyapunov exponent can be shown to be also $1/3$. \square

Remark 4.7. The representation, although not induced via a Veech covering from genus 2, is not very far from the representation $\rho_{\mathbf{L}_{2,2},2}$. More precisely, $\rho_{\mathbf{L}_{2,2},2}$ is taken to $\rho_{\mathbf{M},4}$ by the orientation-reversing outer automorphism

$$\alpha : (T^2, S) \mapsto (T^{-2}S, S)$$

of Γ_Θ , i. e. $\rho_{\mathbf{L}_{2,2},2} \circ \alpha = \rho_{\mathbf{M},4}$.

5. MODULAR EMBEDDINGS OF RANK 2

As we have seen in Section 2, a variation of Hodge structures can equally well be described as a group homomorphism ρ plus the ρ -equivariant period map. In this section, we study these objects from an abstract point of view and we exhibit rigidity properties.

In the following, let $G = \text{PSL}_2(\mathbb{R})$.

Definition 5.1. A *modular embedding* (of rank 2 and weight 1) is a pair (p, ρ) , where

- (i) $\rho : \Gamma \rightarrow G$ is a group homomorphism from a lattice $\Gamma \leq G$.
- (ii) $p : \mathbb{H} \rightarrow \mathbb{H}$ is a ρ -equivariant holomorphic map.

Definition 5.2. For a modular embedding (p, ρ) , denote by $\text{dom}(\rho)$ the domain and by $\text{im}(\rho)$ the image of ρ . We call a modular embedding *discrete* if $\text{im}(\rho)$ is a discrete subgroup of G , and *cofinite* if $\text{im}(\rho)$ acts discretely and cofinitely on \mathbb{H} . If $\text{im}(\rho)$ acts cofinitely, then p descends to a holomorphic map between the quotients, which we denote by \bar{p} .

Note also that we allow Γ to contain torsion elements in order to handle the orbifold case.

Remark 5.3. As in the proof of Theorem 1.1, one shows that if a modular embedding is discrete, then it is either constant (i. e. p is constant) or cofinite.

Examples of modular embeddings come from Teichmüller curves. Apart from the examples given above, there are prominent ones in \mathcal{M}_2 discovered in [McM03]. Here, $\mathrm{SL}(X, \omega)$ injects into $\mathrm{SL}_2(\mathfrak{o}_D)$ for some order \mathfrak{o}_D in a totally real quadratic number field $\mathbb{Q}(\sqrt{D})$. The VHS splits into two sub-VHS of rank 2, and the period map of the non-uniformizing sub-VHS, together with the representation of $\mathrm{SL}(X, \omega) \cong \mathrm{Aff}(X, \omega)$ given by Galois conjugation give rise to a modular embedding. Other examples related to these are the twisted Teichmüller curves studied in [Wei12].

5.1. Rigidity. In this section, we gather results on how much the different data of a modular embedding determine each other.

If p is non-constant, it is easy to see that the representation of a modular embedding (p, ρ) is uniquely determined by p . Conversely, the period map is also uniquely determined by the representation. This has already been remarked by McMullen [McM03, Section 10], and can in fact be generalized to ball quotients [KM12, Theorem 5.4]. We recall the arguments for the convenience of the reader.

Proposition 5.4. *Given a non-trivial group homomorphism $\rho : \Gamma \rightarrow G$ from a cofinite Fuchsian group Γ , there exists at most one map $p : \mathbb{H} \rightarrow \mathbb{H}$ such that (p, ρ) is a modular embedding.*

Proof. We work in the unit disk model and use arguments displayed in [Shi04, Section 2]. Since Γ is a lattice, it is of divergence type. Therefore the set of points E in $\partial\mathbb{D} = S^1$, which can be approximated by a sequence $(\gamma_k(x_0))_k \subset \Gamma$ (for some $x_0 \in \mathbb{D}$) that stays in an angular sector, is of full Lebesgue-measure in $\mathbb{R} \cup \{\infty\}$. For a holomorphic map $p : \mathbb{D} \rightarrow \mathbb{D}$ define $p^*(\zeta)$ of a point of approximation $\zeta \in E$ by $\lim_k p(\gamma_k(x_0))$ for a sequence $\gamma_k(x_0) \rightarrow \zeta$. This is well-defined for almost all ζ and $p^*(\zeta) \in \partial\mathbb{D}$ for almost all ζ by [Shi04, Lemma 2.2].

Now suppose we are given two ρ -equivariant maps p_i , $i = 1, 2$. Pick a point $x_0 \in \mathbb{D}$. If p_1 is constant then $\rho(\Gamma)$ lies in the stabilizer of $p_1(x_0)$. By equivariance, $p_2(y)$ is stabilized by $\rho(\Gamma)$ for any $y \in \mathbb{D}$. Since ρ is non-trivial, $p_1 = p_2$. Thus we are left with the case that p_1, p_2 are non-constant. Then for all k

$$d_{\mathbb{D}}(p_1(x_0), p_2(x_0)) = d_{\mathbb{D}}(p_1(\gamma_k x_0), p_2(\gamma_k x_0))$$

and since $p_i(\gamma_k x_0) \rightarrow \partial\mathbb{D}$, this means that $p_1^*(\zeta) = p_2^*(\zeta)$ for ζ in a set of full measure of $\partial\mathbb{D}$. Thus $(p_1 - p_2)^* = p_1^* - p_2^* \equiv 0$ and therefore $p_1 = p_2$. \square

If (p, ρ) is cofinite, then it determines a map $\bar{p} : \mathbb{H} / \mathrm{dom}(\rho) \rightarrow \mathbb{H} / \mathrm{im}(\rho)$. Conversely, a map \bar{p} between the quotients gives rise to a modular embedding as the following lemma shows. Thus there are in some sense many modular embeddings.

Lemma 5.5. *Let $\bar{p} : \mathbb{H} / \Gamma \rightarrow \mathbb{H} / \Delta$ be a non-constant holomorphic map between finite-area Riemann surfaces. Denote $u_{\Delta} : \mathbb{H} \rightarrow \mathbb{H} / \Delta$ the canonical projection, and let $z \in \bar{p}(\mathbb{H} / \Gamma)$. If $\Delta \subset G$ acts freely on \mathbb{H} , then \bar{p} lifts to a holomorphic map $p : \mathbb{H} \rightarrow \mathbb{H}$, unique up to the choice of a point $\tilde{z} \in u^{-1}(z)$, and there is a unique group homomorphism*

$$\rho : \Gamma \rightarrow \Delta$$

such that p is ρ -equivariant.

We suspect that this statement is well-known, but we are not aware of a source. We supply a proof for the convenience of the reader.

Proof. The first claim follows from u_Δ being a covering map. As to the second claim, note that Δ acts freely and transitively on $u_\Delta^{-1}(z) = \Delta \cdot \tilde{z}$. Let $y \in \bar{p}^{-1}(z)$, and let $\tilde{y} \in u_\Gamma^{-1}(y)$ with $p(\tilde{y}) = \tilde{z}$. (This fixes p .) For any $\gamma \in \Gamma$, there is by assumption a unique $\delta_{\gamma, \tilde{y}} \in \Delta$ such that

$$p(\gamma \tilde{y}) = \delta_{\gamma, \tilde{y}} p(\tilde{y}).$$

Thus we can define a map $\rho : \Gamma \rightarrow \Delta$, $\rho(\gamma) = \delta_{\gamma, \tilde{y}}$. To check that ρ is a group homomorphism, we first show that if $c : [0, 1] \rightarrow \mathbb{H}$ is a path starting at \tilde{y} , then $p(\gamma c(t)) = \delta_{\gamma, \tilde{y}} p(c(t))$ for all t . We know that for each t there is $\delta_{\gamma, c(t)}$ such that $p(\gamma c(t)) = \delta_{\gamma, c(t)} p(c(t))$. On the other hand, we claim that the assignment $t \rightarrow \delta_{\gamma, c(t)}$ is locally constant, and hence constant since $[0, 1]$ is connected: Each $\tilde{x} \in \mathbb{H}$ has a neighborhood U such that for all $\tilde{w} \in U$, $p(\gamma \tilde{w}) = \delta' p(\tilde{w})$ only holds for $\delta' = \delta_{\gamma, \tilde{x}}$. Indeed, it suffices to take $U = p^{-1}(V)$, where V is a neighborhood of $p(\tilde{x})$ such that $\delta V \cap V = \emptyset$ for all $\delta \in \Delta$, $\delta \neq \text{id}$.

This shows that $p(\gamma \gamma' \tilde{y}) = \delta_{\gamma, \tilde{y}} p(\gamma' \tilde{y})$ for $\gamma, \gamma' \in \Gamma$: we take c to be a path connecting \tilde{y} and $\gamma' \tilde{y}$. Thus we have

$$p(\gamma \gamma' \tilde{y}) = \delta_{\gamma, \tilde{y}} p(\gamma' \tilde{y}) = \delta_{\gamma, \tilde{y}} \delta_{\gamma', \tilde{y}} p(\tilde{y}) = \delta_{\gamma \gamma', \tilde{y}} p(\tilde{y})$$

by uniqueness. The uniqueness of ρ follows directly from the construction. \square

Remark 5.6. Given a modular embedding, we can consider the case when one of the two items is an isomorphism. If $p = A \in G$ is a Möbius transformation, then clearly ρ is conjugation by A . Conversely, suppose ρ is an isomorphism. If (p, ρ) is cofinite, then after passing to a finite index subgroup, we can suppose that Γ is torsionfree and that $g(\mathbb{H}/\Gamma) > 1$. Then $\bar{p} : \mathbb{H}/\Gamma \rightarrow \mathbb{H}/\rho(\Gamma)$ must have degree 1 by the Riemann-Hurwitz formula, and hence p is an isomorphism. If (p, ρ) is not cofinite, it may well happen however that ρ is an isomorphism without p being one. Examples are provided by Teichmüller curves in $g = 2$ for non-square discriminants where ρ is induced by Galois conjugation, but p is not an isometry (see [McM03, Theorem 4.2]).

Using the previous lemma, we can now pick up the discussion from the introduction and show that every rational number in $[0, 1]$ is the Lyapunov exponent of a family of elliptic curves.

Proposition 5.7. *For any rational number $0 \leq \lambda \leq 1$, there is a family of elliptic curves $\phi : \mathcal{X} \rightarrow \mathbb{H}/\Delta$ such that λ is in the Lyapunov spectrum of its VHS.*

Proof. Let $\Gamma(2) = \ker(\text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/(2)))$ and let $P\Gamma(2)$ be its projection to $\text{PSL}_2(\mathbb{R})$. We construct a holomorphic map

$$\bar{p} : X \rightarrow \mathbb{H}/P\Gamma(2) \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

of degree d from a Riemann surface X by specifying a monodromy. The map p should be ramified over the cusps and over r interior points x_1, \dots, x_r in such a way that the associated covering is connected and $|\bar{p}^{-1}(x_i)| = t_i$. We can surely find such a monodromy

$$\sigma : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty, x_1, \dots, x_r\}) \rightarrow S_d,$$

since the fundamental group is free of rank $r + 2$ (to guarantee connectedness, we can take \bar{p} to be totally ramified over ∞).

Next we choose a lattice $\Delta \leq \mathrm{PSL}_2(\mathbb{R})$ such that $X \cong \mathbb{H}/\Delta$. Since $P\Gamma(2)$ is torsionfree, we obtain a group homomorphism $\rho : \Delta \rightarrow P\Gamma(2)$ by Lemma 5.5. We can lift this homomorphism to $\tilde{\rho} : \Delta \rightarrow \Gamma(2)^+$, where $\Gamma(2)^+$ is the group generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, an index 2 subgroup in $\Gamma(2)$: for $a \in \Delta$ we let $\tilde{\rho}(a)$ be the unique lift of $\rho(a)$ to $\mathrm{SL}_2(\mathbb{R})$ that is in $\Gamma(2)^+$.

Let p be the lift of \bar{p} . The pair $(p, \tilde{\rho})$ is then a modular embedding, and the associated VHS is the VHS of a family $\phi : \mathcal{X} \rightarrow \mathbb{H}/\Delta$ of elliptic curves. In fact, f is the pullback via p of the universal family over \mathbb{H} . By Proposition 3.1, its sole non-negative Lyapunov exponent is given by

$$\lambda = \frac{\deg(\tilde{\rho}) \mathrm{vol}(\mathbb{H}/\Gamma(2))}{\mathrm{vol}(\mathbb{H}/\Delta)} = \frac{\deg(\tilde{\rho}) \chi(\mathbb{H}/\Gamma(2))}{\chi(\mathbb{H}/\Delta)}.$$

We have $\chi(\mathbb{H}/\Gamma(2)) = -1$ and $\chi(\mathbb{H}/\Delta)$ and $d = \deg(p)$ are related by the Riemann-Hurwitz formula

$$-\chi(\mathbb{H}/\Delta) = 2g(\mathbb{H}/\Delta) - 2 + s(\Delta) = d + \sum_{i=1}^r d - t_i = d(r+1) - \sum_{i=1}^r t_i,$$

where $s(\Delta)$ is the number of cusps of Δ . Therefore,

$$\lambda = \left(r + 1 - \frac{\sum_i t_i}{d} \right)^{-1}$$

where $t_i \in \{1, \dots, d\}$. For fixed r, d , the possible values of λ^{-1} are

$$\left\{ r + 1 - \frac{l}{d} \mid l = r, r+1, \dots, rd \right\}.$$

Letting r and d vary, we can thus obtain every rational number ≥ 1 . Hence every $\lambda \in \mathbb{Q} \cap (0, 1]$ can be realized as Lyapunov exponent. Finally, $\lambda = 0$ is the Lyapunov exponent of a constant family of elliptic curves. \square

5.2. Commensurability and Lyapunov exponents. We define (weak) commensurability of two modular embeddings and show that the Lyapunov exponent of a modular embedding is a weak commensurability invariant. Further, we define the commensurator $\mathrm{Comm}(p, \rho)$ of a modular embedding in analogy to the usual commensurator. If ρ has a non-trivial kernel, we show that $\mathrm{dom}(\rho)$ is of finite-index in $\mathrm{Comm}(p, \rho)$.

Definition 5.8. Two period data (p_i, ρ_i) , $i = 1, 2$ are *commensurable* if there exists $\Gamma' \leq G$, which is a subgroup of finite index in $\Gamma_i = \mathrm{dom}(\rho_i)$ for $i = 1, 2$, such that $\rho_1 = \rho_2$ on Γ' .

As is easily seen, commensurability is an equivalence relation. Note that by Proposition 5.4, $p_1 = p_2$ for commensurable period data.

There is a left action of $G \times G$ on modular embeddings. For $(g, h) \in G \times G$ and a modular embedding (p, ρ) ,

$$g(p, \rho)h^{-1} := (g \circ p \circ h^{-1}, c_g \circ \rho \circ c_{h^{-1}})$$

Definition 5.9. We call two modular embeddings (p_i, ρ_i) , $(i = 1, 2)$ *weakly commensurable* if they become commensurable under this action, i.e. if there exist $(g, h) \in G \times G$ and Γ' , a subgroup of finite index in both Γ_1 and $h\Gamma_2h^{-1}$, such that $\rho_1 = c_g \circ \rho_2 \circ c_{h^{-1}}$ on Γ' .

Example 5.10. Clearly, the two modular embeddings from Section 4 are not commensurable, since otherwise they would agree on $\langle T^{2m} \rangle$ for some $m \in \mathbb{N}$, but $\rho_{\mathbf{L}_{2,2,2}}(T^2)$ is parabolic whereas $\rho_{\mathbf{M},4}(T^2)$ is elliptic.

Moreover, for no two matrices $(g, h) \in \mathrm{SL}_2(\mathbb{Z})^2$ is $g(\rho_{\mathbf{L}_{2,2,2}}, \rho_{\mathbf{L}_{2,2,2}})h^{-1}$ commensurable with $(\rho_{\mathbf{M},4}, \rho_{\mathbf{M},4})$, since conjugation by h cannot exchange the cusps, as they are of different width, and conjugation by g preserves the type (parabolic, respectively elliptic) of the image of a parabolic.

It remains to decide whether the two modular embeddings are not weakly commensurable, and more generally whether $(\rho_{\mathbf{M},4}, \rho_{\mathbf{M},4})$ is (weakly) commensurable to a non-uniformizing representation of an arithmetic Teichmüller curve in $\Omega\mathcal{M}_2(2)$ (i.e. one generated by a square-tiled surface). However, we can exclude that $\rho_{\mathbf{M},4}$ is weakly commensurable to a non-uniformizing representation of an arithmetic Teichmüller curve in $\Omega\mathcal{M}_2(1, 1)$ as the following discussion shows.

Definition 5.11. If (p, ρ) is a cofinite modular embedding, we define its Lyapunov exponent to be

$$\lambda(p, \rho) = \frac{\deg(\bar{p}) \mathrm{vol}(\mathbb{H} / \mathrm{im}(\rho))}{\mathrm{vol}(\mathbb{H} / \mathrm{dom}(\rho))}.$$

This definition is justified by Theorem 1.1 in that if ρ admits a lift to $\mathrm{SL}_2(\mathbb{R})$ and if $\mathrm{dom}(\rho)$ acts freely, we obtain a VHS with Lyapunov exponent $\lambda(p, \rho)$.

Proposition 5.12. *The Lyapunov exponent of a modular embedding is a weak commensurability invariant.*

Proof. The value of $\lambda(p, \rho)$ clearly remains unchanged under the $G \times G$ -action and under passage to a finite-index subgroup by Lemma 3.2. \square

Definition 5.13. For a modular embedding, we define the *commensurator*

$$\mathrm{Comm}(p, \rho) := \{(g, h) \in G \times G \mid (p, \rho), g(p, \rho)h^{-1} \text{ are commensurable}\}.$$

Remark 5.14. $\mathrm{Comm}(p, \rho)$ is a group containing $\Gamma = \mathrm{dom}(\rho)$ via $\gamma \mapsto (\gamma, \rho(\gamma))$. Moreover, for two modular embeddings (p_i, ρ_i) , $i = 1, 2$ that are commensurable, we have $\mathrm{Comm}(p_1, \rho_1) = \mathrm{Comm}(p_2, \rho_2)$.

Further, $\mathrm{Comm}(p, \rho)$ maps into the commensurator

$$\mathrm{Comm}(\mathrm{dom}(\rho)) = \{h \in G \mid h \mathrm{dom}(\rho) h^{-1}, \mathrm{dom}(\rho) \text{ are commensurable}\}$$

by $(g, h) \mapsto h$, and this map is injective if p is not constant. If p is not constant, we can therefore consider $\mathrm{Comm}(p, \rho)$ as a subgroup of G . In fact, it maps into $G_p = \{h \in G \mid \exists g \in G : g \circ p = p \circ h\}$ by rigidity.

As in the case of the usual commensurator, we have the following dichotomy. This is proved verbatim as in e.g. [Zim84, Prop. 6.2.3].

Proposition 5.15. *$\mathrm{Comm}(p, \rho)$ is either dense in G or $\Gamma \leq \mathrm{Comm}(p, \rho)$ is a subgroup of finite index.*

Proposition 5.16. *Suppose we are given a modular embedding (p, ρ) such that p is non-constant and ρ has a nontrivial kernel. Then $\Gamma \leq \mathrm{Comm}(p, \rho)$ is of finite index.*

Proof. Assume $\mathrm{Comm}(p, \rho)$ is dense in G . We claim that $G_p = G$. For let $h \in G$, and let $h_n \in \mathrm{Comm}(p, \rho)$ be a sequence such that $h_n \rightarrow h$ for $n \rightarrow \infty$ in the Hausdorff topology of G . For each h_n there is $g_n \in G$ such that $g_n \circ p = p \circ h_n$. We

claim that $(g_n)_n$ converges to $g \in G$. We show that g_n is a Cauchy sequence, i. e. for all $\varepsilon > 0$ there is n_0 such that for all $n, m > n_0$ and $z \in \mathbb{H}$, $d_{\mathbb{H}}(g_n z, g_m z) < \varepsilon$. It suffices to show this only for all z in some open subset of \mathbb{H} , e. g. in $p(\mathbb{H})$. Then by the Schwarz-Pick lemma,

$$d_{\mathbb{H}}(g_n z, g_m z) = d_{\mathbb{H}}(g_n p(w), g_m p(w)) \leq d_{\mathbb{H}}(h_n w, h_m w) \rightarrow 0$$

uniformly in $w \in \mathbb{H}$. Thus $g_n \rightarrow g$, and $gp(z) = \lim_n g_n(p(z)) = \lim_n p(h_n(z)) = p(\lim_n h_n(z)) = p(hz)$ by continuity of p , and $h \in G_p$.

If $G_p = G$, then ρ admits an extension $\rho' : G \rightarrow G$ by definition of G_p . But then $\ker(\rho')$ is a nontrivial, proper normal subgroup of G , contradicting the fact that G is simple. \square

Remark 5.17. In both our examples of Section 4, there is a nontrivial kernel. Thus we can apply Proposition 5.16. Since in both cases, $\deg(\bar{p}) = 1$ and the image group is $\mathrm{SL}_2(\mathbb{Z})$, which is finitely maximal (i. e. it is not properly contained in a bigger Fuchsian group), we find that the commensurator of $(\rho_{\mathbf{L}_{2,2,2}}, p_{\mathbf{L}_{2,2,2}})$, respectively $(\rho_{\mathbf{M}_{4,4}}, p_{\mathbf{M}_{4,4}})$ coincides with Γ_{Θ} .

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